

# TIME-REVERSAL METHODS FOR ACOUSTO-ELASTIC EQUATIONS AND APPLICATIONS

**Franck Assous\* and Moshe Lin\***

\* Department of Mathematics

Ariel University,

Ariel, 40700, Israel.

e-mail: [franckassous55@gmail.com](mailto:franckassous55@gmail.com)

e-mail: [moshelin1@walla.co.il](mailto:moshelin1@walla.co.il)

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**Abstract.** Time reversal (TR) is a subject of very active research. The principle is to take advantage of the reversibility of wave propagation phenomena, for example in acoustics, elastic or electromagnetism in an unknown medium, to back-propagate signals to the sources that emitted them. In a previous paper [1], we introduced a time-reversed method for acoustic equation. In this paper, our aim is to extend this approach to elastodynamics equations. As the application we have in mind are concerned with ultrasound-based elasticity imaging methods, we consider both elastic and acousto-elastic systems of equations. We stress that our method does not rely on any a priori knowledge of the physical properties of the inclusion.

## 1 INTRODUCTION

Time reversal (TR) is a subject of very active research. The principle is to take advantage of the reversibility of wave propagation phenomena, for example in acoustics,

elastic or electromagnetism in an unknown medium, to back-propagate signals to the sources that emitted them. The initial experiment [2], was to refocus, very precisely, a recorded signal after passing through a barrier consisting of randomly distributed metal rods. Since then, numerous applications of this physical principle have been designed, for instance [3] and references therein. The first mathematical analysis can be found in [4] for a homogeneous medium and in [5], [6] for a random medium.

In a previous paper [1], we introduced a time-reversed method named TRAC (Time-Reversed Absorbing Conditions) for acoustic equation. This method enables one to recreate the past without knowing the location and the properties of the inclusion which diffracted the signals that are back-propagated. This was made possible by removing a small region surrounding the scattering inclusion.

In this paper, our aim is to extend our approach to elastodynamics equations. As the application we have in mind are concerned with ultrasound-based elasticity imaging methods, we also consider a coupled acousto-elastic system of equations, the elastodynamics case being a particular case of the acousto-elastic one. For the sake of simplicity, fluid-solid media that we choose to be "layered". Remark that the method does not require *a priori* knowledge of the physical properties of the inclusion: hard, soft and penetrable inclusions can be treated in the same way.

## 2 FORWARD PROBLEM

We first formulate the mathematical forward model we consider. As we are concerned with the elastic wave or acousto-elastic wave equation, we will present here the more complete one, that is the second one. The elastic governing equations can be easily derived by removing the fluid part (and related equations) from the model described below.

We consider a two-dimensional fluid-elastic domain  $\Omega$  made of two parts, an acoustic one  $\Omega_f$  and an elastic one  $\Omega_s$ . For simplicity, we will assume that  $\Omega$  is a rectangle. The acoustic part of the domain  $\Omega_f$  corresponds to a fluid homogeneous part, characterized

by its density  $\rho_f$  and its Lamé parameter  $\lambda_f$ . We denote by  $\partial\Omega_f$  the boundary of  $\Omega_f$  and  $\mathbf{n}$  is the outward unit normal to the boundary. Introduce the pressure  $p(\mathbf{x}, t)$  on a time  $t$ ,  $\mathbf{x} = (x_1, x_2) \in \Omega_f$ , and  $f(\mathbf{x}, t)$  is a given source, for instance a Ricker function, the acoustic wave equation in  $\Omega_f$  is written

$$\frac{1}{\lambda_f} \frac{\partial^2 p}{\partial t^2} - \operatorname{div} \left( \frac{1}{\rho_f} \nabla p \right) = f, \quad (1)$$

together with homogeneous initial conditions at the initial time  $t = 0$

$$p(t = 0) = 0, \quad \frac{\partial p}{\partial t}(t = 0) = 0. \quad (2)$$

We assume that the boundary  $\partial\Omega_f$  can be split in  $\partial\Omega_f = \Gamma_f \cup \Gamma_I$ , where  $\Gamma_I$  denotes the interface between the fluid and elastic part, assumed, for simplicity reasons, to be horizontal. We supplement the system with absorbing boundary conditions [7] on  $\partial\Omega_f$ . Denoting by  $V_p = \sqrt{\frac{\lambda}{\rho}}$  the wave velocity in the fluid, we get

$$\frac{\partial p}{\partial t} = -V_p \nabla p \cdot \mathbf{n} \quad \text{on } \Gamma_f. \quad (3)$$

On the part  $\Gamma_I$ , we add an interface condition for the pressure  $p(\mathbf{x}, t)$ , that will be presented below, see (9).

We then introduce the governing equations of linear elastodynamics for the elastic part of the domain  $\Omega_s$ , characterized by the density  $\rho_s$ , and the Lamé parameter  $\lambda_s$  and  $\mu_s$ . We assume that the boundary  $\partial\Omega_s$  can be split in  $\partial\Omega_s = \Gamma_s \cup \Gamma_I$ . Denoting by  $\mathbf{u}(\mathbf{x}, t) = (u_1(x_1, x_2, t), u_2(x_1, x_2, t))$  the velocity on a time  $t$ , at a point  $\mathbf{x} = (x_1, x_2) \in \Omega_s$ , we have

$$\rho_s \frac{\partial^2 u_1}{\partial t^2} - \frac{\partial}{\partial x_1} \left( (\lambda_s + 2\mu_s) \frac{\partial u_1}{\partial x_1} + \lambda_s \frac{\partial u_2}{\partial x_2} \right) - \frac{\partial}{\partial x_2} \left( \mu_s \frac{\partial u_1}{\partial x_2} + \mu_s \frac{\partial u_2}{\partial x_1} \right) = 0, \quad (4)$$

$$\rho_s \frac{\partial^2 u_2}{\partial t^2} - \frac{\partial}{\partial x_1} \left( \mu_s \frac{\partial u_1}{\partial x_2} + \mu_s \frac{\partial u_2}{\partial x_1} \right) - \frac{\partial}{\partial x_2} \left( (\lambda_s + 2\mu_s) \frac{\partial u_2}{\partial x_2} + \lambda_s \frac{\partial u_1}{\partial x_1} \right) = 0. \quad (5)$$

The above equations can be written also in the following compact form

$$\rho_s \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot (\mu_s \nabla \mathbf{u}) - \nabla((\lambda_s + \mu_s) \nabla \cdot \mathbf{u}) = 0, \quad (6)$$

together with homogeneous initial conditions:

$$\mathbf{u}(t=0) = 0, \quad \frac{\partial \mathbf{u}}{\partial t}(t=0) = 0. \quad (7)$$

These equations are also supplemented with the absorbing boundary conditions on  $\Gamma_s$ , as proposed in [8]. Introduce the following matrix  $A$ :

- $A = \begin{pmatrix} -\sqrt{\rho_s(\lambda_s + 2\mu_s)} & 0 \\ 0 & -\sqrt{\rho_s\mu_s} \end{pmatrix}$  for vertical boundary edges,
- $A = \begin{pmatrix} -\sqrt{\rho_s\mu_s} & 0 \\ 0 & -\sqrt{\rho_s(\lambda_s + 2\mu_s)} \end{pmatrix}$  for horizontal boundary edges,

and the  $(2 \times 2)$  matrix  $T := (\tau_{ij})_{1 \leq i, j \leq 2}$  defined by

- $\tau_{11} = \lambda_s \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu_s \frac{\partial u_1}{\partial x_1}$
- $\tau_{12} = \tau_{21} = \mu_s \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right)$
- $\tau_{22} = \lambda_s \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} \right) + 2\mu_s \frac{\partial u_2}{\partial x_2}$

$$A \frac{\partial}{\partial t} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \tau_{ij} \cdot \mathbf{n} \quad (8)$$

Finally, we introduce the transmission conditions at the interface  $\Gamma_I$  between the fluid and the solid parts, the pressure-velocity formulation allowing us to express them easily.

We set

$$\frac{1}{\rho_f} \frac{\partial p}{\partial x_2} = -\frac{\partial u_2}{\partial t} \quad (9)$$

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 0, \quad \frac{\partial p}{\partial t} = \lambda_s \frac{\partial u_1}{\partial x_1} + (\lambda_s + 2\mu_s) \frac{\partial u_2}{\partial x_2} \quad (10)$$

In particular, these conditions will appear naturally in the variational formulation, that will be basis of the finite element method.

### 3 TIME REVERSE PROBLEMS

In a second step, we will use time reverse methods to solve time reversed acousto-elastic and elastodynamics problems. Recall that the main idea of time reverse is to take advantage of the reversibility of wave propagation phenomena in a non dissipative unknown medium to back-propagate signals to the sources that emitted them [9]. Examples for time reverse technique (numerical or experimental) are presented in [2, 9, 10, 11].

We first introduce the time-reversed wave equations for the acoustic part of the domain  $\Omega_f$ . We denote by  $p^R(\mathbf{x}, t')$  the time-reversed pressure, defined by  $p^R(\mathbf{x}, t') = p(\mathbf{x}, T - t)$ ,  $\mathbf{x} \in \Omega_f$ , where  $T$  denotes the final time. Since the wave equation involves only second order time derivatives, this definition ensures that the reverse field  $p^R(\mathbf{x}, t')$  is a solution to the wave equation

$$\frac{1}{\lambda_f} \frac{\partial^2 p^R}{\partial t'^2} - \operatorname{div} \left( \frac{1}{\rho_f} \nabla p^R \right) = 0, \quad (11)$$

together with initial conditions:

$$p^R(t' = 0) = p(t = T), \quad \frac{\partial p^R}{\partial t'}(t' = 0) = \frac{\partial p}{\partial t}(t = T), \quad (12)$$

with the TR absorbing boundary conditions on  $\Gamma_f$ , analogous to (3).

Similarly, we also introduce the elastic time-reversed problem associated to equations (4-8). We denote by  $\mathbf{u}^R(\mathbf{x}, t') = (u_1^R(x_1, x_2, t'), u_2^R(x_1, x_2, t'))$  the time-reversed velocity solution to linear elastodynamics, that solves

$$\rho_s \frac{\partial^2 u_1^R}{\partial t'^2} - \frac{\partial}{\partial x_1} ((\lambda_s + 2\mu_s) \frac{\partial u_1^R}{\partial x_1} + \lambda_s \frac{\partial u_2^R}{\partial x_2}) - \frac{\partial}{\partial x_2} (\mu_s \frac{\partial u_1^R}{\partial x_2} + \mu_s \frac{\partial u_2^R}{\partial x_1}) = 0, \quad (13)$$

$$\rho_s \frac{\partial^2 u_2^R}{\partial t'^2} - \frac{\partial}{\partial x_1} (\mu_s \frac{\partial u_1^R}{\partial x_2} + \mu_s \frac{\partial u_2^R}{\partial x_1}) - \frac{\partial}{\partial x_2} ((\lambda_s + 2\mu_s) \frac{\partial u_2^R}{\partial x_2} + \lambda_s \frac{\partial u_1^R}{\partial x_1}) = 0, \quad (14)$$

together with initial conditions

$$\mathbf{u}^R(t' = 0) = \mathbf{u}(t = T), \quad \frac{\partial \mathbf{u}^R}{\partial t'}(t' = 0) = \frac{\partial \mathbf{u}}{\partial t}(t = T). \quad (15)$$

The absorbing boundary conditions on  $\Gamma_s$  have the same expression as in (8) simply by replacing  $\mathbf{u}$  by  $\mathbf{u}^R$ .

Finally, we derive the time-reversed continuity transmission conditions at the interface  $\Gamma_I$  that can be written

$$\frac{1}{\rho_f} \frac{\partial p^R}{\partial x_2} = -\frac{\partial u_2^R}{\partial t'} \quad (16)$$

$$\frac{\partial u_1^R}{\partial x_2} + \frac{\partial u_2^R}{\partial x_1} = 0, \quad \frac{\partial p^R}{\partial t'} = \lambda_s \frac{\partial u_1^R}{\partial x_1} + (\lambda_s + 2\mu_s) \frac{\partial u_2^R}{\partial x_2} \quad (17)$$

In order to create synthetic data, the forward and reverse formulations are approximated by the FreeFem++ package [14] which implements a finite element method in space. In this study we use a standard  $P^2$  finite element method. The advancement in time is given by a second order central finite difference scheme so that it is time reversible also on the numerical level.

## 4 NUMERICAL RESULTS

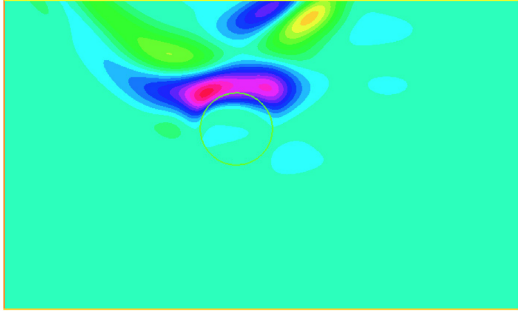
In this section, we describe numerical results obtained for a scatter identification problem, in the case of layered acousto-elastic medium. The principle of the method can be described as follows: let us consider an incident wave impinging on an inclusion  $D$  characterized by different physical properties from the surrounding medium. This incident wave is generated by a point source such that after a time  $T_f$  the total field is negligible. We introduce a boundary - modeling source-receivers array (SRA) - where the forward signal is recorded.

Then, we perform numerically a time-reversed computation, by back propagating the recorded data from the SRA. However, we do not assume we know the physical properties of the inclusion or its location. Hence, the recorded data are back propagated in the medium without the inclusion  $D$ . Finally, we intend to image the unknown scatterer in the medium - responsible of the diffraction of the incident wave - by using correlation method between the forward and the reversed wave, in the same spirit as those involved

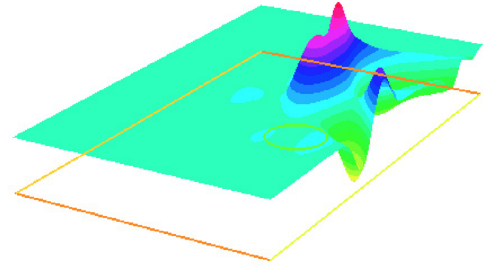
for instance in time reverse migration [12].

To illustrate our purpose, we consider a two layered medium, made of fluid part (top) and of a elastic one (bottom). The source  $f(\mathbf{x}, t)$  and the SRA are located in the homogeneous fluid part of the medium, whereas the inclusion  $D$  is located in the elastic part, assumed to be an homogeneous and isotropic medium with propagation velocities  $V_p$  and  $V_s$ . Hence, the scatterer  $D$  is illuminated by an incident acoustic field, that is first transmitted to the elastic medium through the interface  $\Gamma_I$ , and then scattered by the inclusion  $D$ , before to be recorded by the SRA. The SRA being located in the fluid part, they are able to record only a scalar quantity (the pressure  $p(\mathbf{x}, t)$ ), and not a vector velocities  $\mathbf{u}(\mathbf{x}, t)$ .

However, as shown on Figure 1, where the correlation image between the forward and the reversed wave is depicted (only in the elastic layer), one is able to determine the existence and location of the inclusion  $D$ . Note also that one can detect the presence of the interface fluid-solid  $\Gamma_I$ .



(a) two-dimensional representation



(b) three-dimensional representation

**Figure 1:** correlation between the forward and the reversed wave (representation of the elastic layer)

## 5 CONCLUSION

In this paper, we proposed a numerical time-reversed approach for elastic or acousto-elastic equations. We have derived a finite element method for both models, implemented

with the Freefem++ software. Preliminary results have been presented to illustrate the feasibility of the algorithm. In particular, the method does not rely on any *a priori* knowledge of the physical properties of the inclusion. In a future paper, other important aspects will also be investigated: more general configurations that mimics a soft tissue, that would lead to (try to) apply our method in the framework of breast tumors detection. In particular, one will consider to use only partial information, or to try to image the scatterer in case of "noisy" (partially unknown) medium. One will also evaluate the quality of the material elasticity parameters obtained by introducing a well adapted cost function, in the same spirit as what is derived for inverse problems. As usual in this context, optimization based algorithm will be necessary to achieve this part.

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